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Bloch's paradox does not appear in quantum mechanics on phase space

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Abstract
It is shown that in the formalism of quantum mechanics on phase space with a purely quantum mechanical theory of measurement, Bloch's paradox does not appear.

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1. Introduction

Bloch's paradox [2], introduced in 1967, stated: 'But problems arise if one imposes two requirements which seem to be quite orthodox: (1) that every particle has, at a given spacetime point, a unique wave function, whether pure state or mixture, which transforms under an irreducible representation of the Lorentz group—or at least each pure state entering the mixture must transform thus; (2) that one and only one component of the mixture in a case like ours is *the* wave function of S [the state of the system being measured] after interaction'. It was named by G Fleming [5]. The paradox may be rephrased as follows[1].

An event occurs in a box in configuration space and time centered at the point $(\overrightarrow{x}, t)$ in relativistic spacetime in one Lorentz frame. In another frame, it occurs in a box centered at the point $(\overrightarrow{x}', t')$. There are an infinite set of frames. So, when exactly does the event occur?

This is exemplified by defining the 'event' to be the 'collapse' of the wavefunction of a particle by its position being measured in a box. Assuming the 'postulate of instantaneous reduction', one then says that for a collapse centered at $(\overrightarrow{x}, t)$, the collapse occurs at time t. We will always assume that collapse occurs with instantaneous reduction, here. Consequently, one may restate Bloch's paradox one of two ways: 'If, in some Lorentz frame, an observable is measured at an instant, then what does the measurement look like in a different frame?' or 'If, in some Lorentz frame, an observable is measured and the state of the system reduces by a projection, then what does the system look like in a different frame?'

[1] The following three paragraphs are a bit vague, but they reflect what I presume is the intent of the original authors.

Aharonov and Albert [1] have discussed this for a massive spinless particle. There, a particle may be in one of three boxes, X_i, $i \in \{1, 2, 3\}$. At time t_1 a measurement is made in X_1 and it is found that the particle is not there. The points in (X_1, t_1) and (X_2, t_2) are separated by a spacelike interval. A Lorentz observer wizzes by for which the events in (X_1, t_1) precede those in (X_2, t_2). Assuming that 'measurements in spacelike-separated regions' necessarily commute, they derive that the wavefunction between t_1 and t_2 is roughly $N(|X_2\rangle + |X_3\rangle)$, N a normalization constant, and where $|X_i\rangle$ represents the wavefunction of the particle in X_i at time t_i, suitably shifted in time so that they may be added. Another observer whips by so that t_2 precedes t_1, the particle is not observed to be in (X_2, t_2), and the observer deduces that the particle in the time between t_1 and t_2 is in the wavefunction $N'(|X_1\rangle + |X_3\rangle)$. That is, the state determined by these two state histories are drastically different! They are so different that they are not relativistic transformations of each other.

See [13] for a review of various authors' examples of different aspects of Bloch's paradox.

We will use the following definitions of 'measurement', 'measurement operator', etc. in Hilbert space $\mathcal{H}$ in what follows:

A state on $\mathcal{H}$ is any operator ρ that is self-adjoint, positive, of trace class with $Tr(\rho) = 1$. An observable is a self-adjoint operator on $\mathcal{H}$ (that may have some other properties to make it 'observable'). Let ρ be a state on $\mathcal{H}$, and A an observable. Then an experimental apparatus designed to 'measure' A will, on a given input state ρ, give a real number. In general the numbers so generated upon repeated measurement of ρ will form a probability distribution. $Tr(\rho A)$ is the expected value of A in state ρ of this distribution. If we vary ρ, we may deduce A from these numbers $Tr(\rho A)$. A is the measurement operator for this experiment. Any self-adjoint operator has a spectral measure associated with it and a spectrum in the reals, or in $\mathbb{R}^n$. If P_ψ denotes the state that is the projection onto the normalized vector $\psi \in \mathcal{H}$, then the transition probability from state P_ψ to state P_φ is just $Tr(P_\psi P_\varphi) = |\langle\psi, \varphi\rangle|^2$ where $\langle\cdot, \cdot\rangle$ is the inner product on the Hilbert space $\mathcal{H}$. If one chooses $\mathcal{H}$ to describe quantum particles, then the vectors in $\mathcal{H}$ all have dispersions in their momentum, $\overrightarrow{p}$, and position, $\overrightarrow{q}$, variables since $\mathcal{H}$ is a vector space of functions that are square integrable over some manifold in $\overrightarrow{p}$ and/or $\overrightarrow{q}$; there are no eigenvectors of either $\overrightarrow{p}$ or $\overrightarrow{q}$.

A measurement operator in a box Δ in the spectrum of A may be taken as the spectral projection onto the (Borel) set Δ. If A is the position operator and one works in a representation in which the functions $\psi \in \mathcal{H}$ are functions of the position, then one may rephrase 'the semi-classical measurement in box Δ at time t of state given by ψ with positive outcome' to imply $\psi(\overrightarrow{x}, t^+) = 0$ if $\overrightarrow{x} \notin \Delta$, $t^+ = \lim_{\varepsilon\to 0, \varepsilon>0}(t + \varepsilon)$. This is a statement in the 'individual' or 'collapse' formalism rather than the statistical formalism of the previous paragraph.

Alternatively, the measurement of state ρ may be the accumulated sum of the transition probabilities to states ϕ for which the $Tr(\phi A)$ is contained in Δ. This alternative definition is closer to the experimental situation. For example, begin with ρ and ask where it appears on a screen composed of certain sites for potential capturing of ρ. Here one may make at most a conditional (statistical) probability distribution for the state of the system after the experiment has been performed.

These two definitions will be termed here the 'semi-classical' (for lack of a better phrase) and the 'quantum' measurement, respectively.

Now, there are some additional unstated assumptions in the first four paragraphs above:

(i) the 'event' or 'collapse' actually is thought to occur (or fails to occur) in a box centered at a point in spacetime;

(ii) a measurement in any of these respective boxes with an outcome other than 'it is not there' causes the wavefunction to collapse to the wavefunction in the corresponding box(es);

(iii) you have a localized theory of measurement;

(iv) you *can* take localized measurements in space-like separated regions.

We make several remarks about these unstated assumptions:

(i.1) One has taken 'measurement' as a semi-classical measurement on the classical spacetime (the (X_i, t_i)) and not as a quantum mechanical interaction between two quantum mechanical objects.

(i.2) The fact that one takes $(\overrightarrow{x}, t)$ as a point in Lorentz/Minkowski space that is associated with (X, t) of an event, when viewed in a different frame, will have the corresponding time component spread out. From this point of view, there seems to be no way that one can have instantaneous reduction consistent with Lorentz/Poincaré invariance.

(i.3) Given that one 'measures' a wavefunction by taking an inner product with a known wavefunction, and as quantum mechanical objects have a spread, one may be unable to claim that the collapse, if a collapse occurs at all, 'occurs in a box' unless it is known somehow that the wavefunction of the test particle is compactly supported in the box. Similarly for non-occurrance.

(i.4) Why in spacetime? One may take the spacetime and enlarge it to a phase spacetime (including spin if necessary). Then one may project it back to configuration spacetime by marginality, if desired. For this to make sense, the measurement operator is taken as a function over phase space and, by marginality, it may be restricted back to configuration space. (More on this later. But this will require that one has a Poincaré invariant theory of measurement in phase space.) That these measurement operators are positive operators and not projections will be seen. Again, one may be lacking any notion of a support in a compact set for the measurement operator.

(ii) Measurement in a box gives a form of von Neumann's collapse postulate. But the collapse postulate is based on the operator-being-measured having a purely discrete spectrum. For operators with purely continuous spectrum, such as the position or momentum operators, von Neumann [12] is silent. Ozawa has shown [7] that in that case the measurement operator cannot be described by a projection operator. It will be shown that in a Poincaré invariant theory of measurement in phase space, only a non-localized description is present and will be such that the von Neumann collapse postulate no longer holds. Furthermore, since the von Neumann collapse postulate is equivalent to the repeatability of a measurement 'immediately after' and giving the same result as the first measurement (a rare event in actuality), we are tempted to replace that postulate with a weakened version of it in which we take the projection operator which describes the measurement and replace it with a positive operator. But even if we replace the projection with just a positive operator, then we do not know that the measurement necessarily leads to a new state with a wavefunction *in the box*. However, if we take either of these views of collapse, and if ρ is a density matrix (state) and our measurement operator is M, a projection or a positive operator, then the state may convert to $NM^{1/2}\rho M^{1/2}$, where N is a normalization constant.

(iii) and (iv) Taking the view that a measurement is in fact a quantum measurement, and that one has an 'informationally complete set of measurements', it will be shown that it is impossible to have measurements in space-like separated regions. Furthermore, it will be shown that the theory of quantum mechanics in phase space [9] gives rise to a theory in which it is impossible to have measurements in space-like separated regions, although the *quantum expectation values of the testing wavefunctions* are in space-like separated regions. All this will be clarified and detailed in what follows. One has a purely non-localized theory that, none-the-less, mimics all the physically verifiable properties predicted by quantum mechanics. The theory that evolves in the following sections is, in fact, another formulation of the theory of quantum mechanics in phase space.

2. A quantum mechanical theory of measurement

One starts with a canonical Hilbert space, $\mathcal{H}$, describing a particle. Suppose that one wishes to test whether a particle in a vector state P_ψ with vector $\psi \in \mathcal{H}$ appears in fact 'at $\vec{x}$ at time t'. (P_ψ is a one-dimensional projection of $\mathcal{H}$ onto the vector ψ.) How can one do this? Well, one could have a potential vector state with wavefunction η and placed so that it has a *quantum expectation value* at $\vec{x}$ at time t. Since quantum states are objects that have dispersion, taking *the quantum expectation value* to be at $\vec{x}$ is the best one can do! Then one may compute the transition probability of ψ with η to 'measure the ψ'. Recall that the transition probability of ψ with η is just $|\langle\psi, \eta\rangle|^2$, where $\langle\cdot, \cdot\rangle$ is the inner product in $\mathcal{H}$.

Note that one has no way of measuring 'strictly locally' since η is not a point mass. But, you say, it never was claimed that one could measure to get a particle exactly at $\vec{x}$ at time t. It would suffice to have η having a support in a (Borel) set Δ, η square integrable, normalized, and expectation value $\vec{x} \in \Delta$. One has just converted from 'measurement at $\vec{x}$ at time t' to a 'localized' measurement within Δ at time t, with the quantum expectation value with respect to η at $\vec{x} \in \Delta$. Next, suppose that the support of η is much smaller than Δ. One could translate η within the (Borel) set Δ, with the translation operator $U(\vec{y})$ to obtain $|\langle\psi, U(\vec{y})\eta\rangle|^2$. Then $U(\vec{y})\eta$ has expectation value $\vec{x} + \vec{y}$, which is taken to be still in Δ. Take the integral over Δ of these transition probabilities as a function of $\vec{y}$ and one will have the transition probability corresponding to a measurement in Δ. But what about when $U(\vec{y})\eta$ is very near the boundary of Δ? If 'very near' means closer than the dispersion of η, then there is a bit of the wavefunction ψ that can overlap with $U(\vec{y})\eta$ even though ψ remains outside of Δ, and even supposing that ψ and η are of compact support. Thus, one is left with a description of Δ quantum mechanically as a sort of fuzzy set function with the fuzziness caused by the dispersion of η. Now, we shall see that to describe an experiment having informational completeness (to be described below), η will necessarily have to have $\langle U(\vec{y})\eta, \eta\rangle$ nonzero for all $\vec{y}$! In other words, there is no theory of localized measurement at all. In fact, taking localized measurements in two spacelike-separated regions is precluded from the start. We still have the quantum expectations may be spacelike separated, however.

Before we leave this theory of measurement given in the first paragraph in this section, we should say that we have here the beginning of the solution to Bloch's paradox. Because the vectors are in $\mathcal{H}$ where $\mathcal{H}$ is an irreducible representation space for which the Poincaré group is unitarily presented, and presuming one has measured a state P_ψ by means of having it transist to state P_η, then upon switching to another frame by $U \in \mathcal{P}$, one gets the transition probability to be

$$Tr(P_{U\psi} P_{U\eta}) = |\langle U\psi, U\eta\rangle|^2 = |\langle\psi, \eta\rangle|^2 = Tr(P_\psi P_\eta); \qquad (2.1)$$

i.e., the same transition probability with which one started.

All the details of the phase space formulation of relativistic quantum mechanics will now be worked out. One starts with the Poincaré (a.k.a. the inhomogeneous Lorentz) group and asks what are the phase (a.k.a. symplectic) spaces on which the Poincaré group acts. By a theorem of Guillemin and Sternberg [6], every one of these phase spaces may be found as a (union over) certain homogeneous space(s) of the Poincaré group; i.e. as $\mathcal{P}/H$, where H is a closed subgroup of the Poincaré group, $\mathcal{P}$. Set $\Gamma = \mathcal{P}/H$. The details of this H are not important for us here, and it is suggested that the reader refers either to [6] or to [9]. However, one takes the space to be a representation space that describes either a massive spinning particle or a massive, spin zero particle. It works out that, $\Gamma = \{(\vec{p}, \vec{q}, R)\}$, with $\vec{p}$ = the momentum, $\vec{q}$ = the position and R = the rotation of spin, or $\Gamma = \{(\vec{p}, \vec{q})\}$ in the spin zero case. (Henceforth we will take the notation of a representation with spin.) On

Γ, we may represent $\mathcal{P}$ by $g : x \mapsto gx, g \in \mathcal{P}, x \in \Gamma$, as Γ is a homogeneous space of $\mathcal{P}$. One takes $\sigma : \Gamma = \mathcal{P}/H \to \mathcal{P}, x \mapsto \sigma(x)$, as a choice function (which may even be chosen to be continuous).

One [9] then takes the Hilbert space of complex-valued, square-integrable functions on Γ, $L^2(\Gamma, \mu)$, with μ equal to the $\mathcal{P}$-left-invariant measure over the phase space Γ. On $L^2(\Gamma, \mu)$, one may represent $\mathcal{P}$ by $[V(g)\Psi](x) = \Psi(g^{-1}x), g \in \mathcal{P}$ (or perhaps as a projective representation). V is unitary. On this space, one may consider the set of operators, $\{A(f)\}$ given by multiplication by (measurable) functions f of the phase space variables. These multiplication operators are covariant under the action of the group $\mathcal{P}$: $V(g)A(f)V(g)^{-1} = A(g.f), g.f(x) = f(g^{-1}x)$.

One also has a canonical Hilbert space, $\mathcal{H}$, describing a particle with momentum, position, mass and spin; i.e., an irreducible representation space for the Poincaré group for a massive, spinning particle. These representation spaces are the well-known quantum mechanical representation spaces. One labels the unitary, irreducible representation of $\mathcal{P}$ on $\mathcal{H}$ by U. Now, picking a normalized $\eta \in \mathcal{H}$, then

$$[W^\eta\phi](x) \equiv \langle U(\sigma(x))\eta, \phi\rangle, \qquad \phi \in \mathcal{H}, \quad x \in \Gamma \tag{2.2}$$

defines $W^\eta\phi$ as a function on Γ. With certain conditions [9] on η, this W^η is a unitary map from $\mathcal{H}$ to a closed subspace of $L^2(\Gamma, \mu)$. In particular, in order for $\{U(\sigma(x))\eta, x \in \Gamma\}$ to be extended to $\{U(g)\eta | g \in \mathcal{P}\}$, η must satisfy

$$\text{(i): } U(h)\eta = \alpha(h)\eta, \qquad h \in H \tag{2.3}$$

such that α is a one-dimensional representation (or *character*) of H. Moreover, if

$$\text{(ii): } \int_\Gamma |\langle U(\sigma(x))\eta, \eta\rangle|^2 \, \mathrm{d}\mu(x) < \infty, \tag{2.4}$$

then one has a situation in which one may reproduce the Hilbert space structure of any $\psi \in \mathcal{H}$ by means of the complex-valued functions on phase space $\langle U(\sigma(x))\eta, \psi\rangle$; i.e., we have a coherent state representation with $\{U(\sigma(x))\eta : x \in \Gamma\}$ as an overcomplete basis of H.

Having complex-valued functions rather than vector-valued functions with vectors in $\mathbb{C}^n$, n having to do with the spin, is remarkable in the case of nonzero spin! [9, page 325]. Another ramification of this representation scheme is that one may enlarge the description of $\mathcal{H}$ as saying that the vectors in $\mathcal{H}$ have dispersions in their positions, momenta, *and spin variables*. This plays no role in the present paper.

Condition (ii) is the 'admissibility' or 'square-integrability' condition, and (i) and (ii) are called 'α-admissibility'. It is found that (i) and (ii) are necessary and sufficient conditions for obtaining a representation of $\mathcal{P}$ on Γ. All the physically relevant irreducible unitary representations in quantum mechanics behave in this way[2]. (See α-admissibility in [9].) With these two conditions on η, one obtains that W^η maps $\mathcal{H}$ to a closed subspace in $L^2(\Gamma, \mu)$ and intertwines the representations U and V: $W^\eta U(g) = V(g)W^\eta, g \in \mathcal{P}$.[3]

Let P^η be the projection from $L^2(\Gamma, \mu)$ to the subspace of $L^2(\Gamma, \mu)$ so obtained. Then one 'pulls back' the operators $A(f)$ on $L^2(\Gamma, \mu)$ to operators $A^\eta(f)$ on $\mathcal{H}$:

$$A^\eta(f) \equiv [W^\eta]^{-1} P^\eta A(f) W^\eta. \tag{2.5}$$

These operators form a set satisfying the ordinary canonical commutation relations, etc. ($A^\eta(f)$ when expanded in the operators for momentum, position, and spin have η appearing only in terms of moments of η.)

[2] We take the 'continuous spin representations' as being non-physical.

[3] The conditions (i) and (ii) also yield the so-called 'orthogonality conditions' which we shall not use here. See [9, III.1.O].

If f is a function with compact support, then $A^\eta(f)$ is a bounded operator. Furthermore, taking η to have quantum expectation values of momentum $= \overrightarrow{0}$, of position $= \overrightarrow{0}$, and of spin $= \overrightarrow{s_0}$, and for $x = (\overrightarrow{p}, \overrightarrow{q}, R)$, where $\overrightarrow{p}$ = boost $\in \mathbb{R}^3$, $\overrightarrow{q}$ = translation $\in \mathbb{R}^3$, and $R =$ rotation, we obtain the quantum expectation values for $U(\sigma(x))\eta$ of $\overrightarrow{p}$ for momentum, $\overrightarrow{q}$ for position, and $R\overrightarrow{s_0}$ for spin[4]. Thus, we may label $\eta_{p,q,R} \equiv U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))\eta$, since $(\overrightarrow{p}, \overrightarrow{q}, R\overrightarrow{s_0})$ comprise the expectation values of (momentum, position, spin) for $U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))\eta$.

Now, a tiny bit of computation will show that

$$\begin{aligned} A^\eta(f) &= \int_\Gamma f(x)T^\eta(x)\,\mathrm{d}\mu(x) \\ &= \int_\Gamma f(\overrightarrow{p}, \overrightarrow{q}, R)T^\eta(\overrightarrow{p}, \overrightarrow{q}, R)\,\mathrm{d}\mu(\overrightarrow{p}, \overrightarrow{q}, R), \end{aligned} \tag{2.6}$$

where

$$T^\eta(x) = U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))P_\eta U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))^\dagger \tag{2.7}$$

is the projection onto $\eta_{p,q,R} \equiv U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))\eta$. Thus, a measurement of $A^\eta(f)$ in a vector state P_ψ gives

$$Tr(P_\psi A^\eta(f)) = \int_\Gamma f(\overrightarrow{p}, \overrightarrow{q}, R)|\langle U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R)\eta, \psi\rangle|^2\,\mathrm{d}\mu(\overrightarrow{p}, \overrightarrow{q}, R); \tag{2.8}$$

i.e., the transition probability from ψ to $U(\sigma(\overrightarrow{p}, \overrightarrow{q}, R))\eta$ integrated over $f(\overrightarrow{p}, \overrightarrow{q}, R)$.

Taking f to be the characteristic function χ_Δ of the set Δ is as close as one can get to a projection onto Δ. (As will be seen, $A^\eta(\chi_\Delta)$ is not a projection!) One may replace χ_Δ with the fuzzy function f $(0 \leqslant f \leqslant 1)$ to obtain a modest generalization. One says that $A^\eta(f)$ provides a measurement in fuzzy set f. Also, note that Δ is a set *in phase space*. To obtain an operator '$A^\eta(\chi_{\Delta'})$' with Δ' a set in configuration space, one has to take $\Delta = X \times \Delta' \times \Omega$ where X is the entire momentum space and Ω is the entire spin space, and then '$A^\eta(\chi_{\Delta'})$' $=$ $A^\eta(\chi_\Delta)$. This amounts to defining '$A^\eta(\chi_{\Delta'})$' from $A^\eta(\chi_\Delta)$ by marginality. More generally, to obtain an operator that depends only on any one variable, integrate out the remaining variables. For a physical justification of having the full phase space in the variables of f, also see [9].

It is pointed out that *measuring these* $A^\eta(f)$ *is the best one can do* for the quantum mechanical measurement of any density operator, including any vector state [4]. In this sense measuring $A^\eta(f)$ s is *optimal* among all the operators that measure the probability of having values in the (fuzzy) set f.

It is stressed that in addition, such measurements provide an informationally complete set, unlike the measurement of just position or just momentum. This will be addressed in the following section.

Next, given a density operator, ρ, one obtains the quantum probability

$$Tr(\rho A^\eta(f)) = \int_\Gamma f(\overrightarrow{p}, \overrightarrow{q}, R)\langle \eta_{p,q,R}, \rho\eta_{p,q,R}\rangle \mathrm{d}\mu(\overrightarrow{p}, \overrightarrow{q}, R). \tag{2.9}$$

Taking $\rho = |\psi\rangle\langle\psi| = P_\psi$ for some $\psi \in \mathcal{H}$, $\|\ \psi\ \| = 1$, one obtains

$$\langle \eta_{p,q,R}, P_\psi \eta_{p,q,R}\rangle = |\langle \eta_{p,q,R}, \psi\rangle|^2, \tag{2.10}$$

the transition probability from ψ to $\eta_{p,q,R}$. If ρ is a general density operator $\sum_i \rho_i P_{\psi_i}$, $\{\psi_i\}$ an orthonormal basis, $\rho_i \in \mathbb{R}_+$, $\sum_i \rho_i = 1$, then $\langle \eta_{p,q,R}, \rho\eta_{p,q,R}\rangle$ is a convex combination of transition probabilities. Consequently, the operator $A^\eta(\chi_\Delta)$ or $A^\eta(f)$ is physically motivated,

4 The cases of mass zero or spin zero are also treated similarly.

and is a measurement operator for the transition to some state that has its expectation values in the box described by χ_Δ or the fuzzy box described by f.

$A^\eta(f)$ has the following additional properties [9]:

Property A: $A^\eta(f)$ is a non-local operator since one is measuring in (2.9) and (2.10) from the *expected values* of $\overrightarrow{p}$, $\overrightarrow{q}$ and $R\overrightarrow{s_0}$ for $\eta_{p,q,R}$ and taking the transition probability from ψ to $\eta_{p,q,R}$ in $\mathcal{H}$.

Property B: For $0 \leqslant f \leqslant 1$, f measurable, $A^\eta(f)$ is a positive operator that is not a projection operator except in the extreme cases $f \equiv 0$ and $f \equiv 1$ [10], but is an 'effect': $\mathbf{0} \leqslant A^\eta(f) \leqslant \mathbf{1}$. This includes the case $f = \chi_\Delta$, so that $A^\eta(\chi_\Delta)$ is not a projection operator for $\Delta \neq \emptyset$ or Γ a.e.μ.

Property C: The set of $A^\eta(f)$ s is covariant with respect to the section of boosts, translations and spin rotations. This may be seen from the left invariance of the measure μ , the α-admissibility of η, and

$$\begin{aligned}
&U(\sigma(\overrightarrow{p},\overrightarrow{q},R))A^\eta(f)U(\sigma(\overrightarrow{p},\overrightarrow{q},R))^\dagger \\
&\quad= \int_\Gamma f(\overrightarrow{p}',\overrightarrow{q}',R')U(\sigma(\overrightarrow{p},\overrightarrow{q},R))T^\eta(\overrightarrow{p}',\overrightarrow{q}',R') \\
&\qquad\times U(\sigma(\overrightarrow{p},\overrightarrow{q},R))^\dagger\,\mathrm{d}\mu(\overrightarrow{p}',\overrightarrow{q}',R') \\
&\quad= \int_\Gamma f(\overrightarrow{p}',\overrightarrow{q}',R')T^\eta(\sigma(\overrightarrow{p},\overrightarrow{q},R)\circ(\overrightarrow{p}',\overrightarrow{q}',R'))\,\mathrm{d}\mu(\overrightarrow{p}',\overrightarrow{q}',R') \\
&\quad= \int_\Gamma f(\sigma(\overrightarrow{p},\overrightarrow{q},R)^{-1}\circ(\overrightarrow{p}'',\overrightarrow{q}'',R''))T^\eta(\overrightarrow{p}'',\overrightarrow{q}'',R'')\,\mathrm{d}\mu(\overrightarrow{p}'',\overrightarrow{q}'',R'') \\
&\quad= A^\eta(\sigma(\overrightarrow{p},\overrightarrow{q},R).f),
\end{aligned} \tag{2.11}$$

where

$$[\sigma(\overrightarrow{p},\overrightarrow{q},R).f](\overrightarrow{p}',\overrightarrow{q}',R') = f(\sigma(\overrightarrow{p},\overrightarrow{q},R)^{-1}\circ(\overrightarrow{p}',\overrightarrow{q}',R')) \tag{2.12}$$

is a (left-regular) representation of the section of boosts, translations and spin changes on functions of the group parameters. Similar results hold for a general Poincaré transformation.

Property D: $T^\eta(\overrightarrow{p},\overrightarrow{q},R)$ are not projections onto the point $(\overrightarrow{p},\overrightarrow{q},R\overrightarrow{s_0})$ as $T^\eta(\overrightarrow{p},\overrightarrow{q},R)T^\eta(\overrightarrow{p}',\overrightarrow{q}',R') \neq 0$ for all $(\overrightarrow{p},\overrightarrow{q},R)$ different from but near $(\overrightarrow{p}',\overrightarrow{q}',R')$. In fact

$$\begin{aligned}
T^\eta(\overrightarrow{p},\overrightarrow{q},R)T^\eta(\overrightarrow{p}',\overrightarrow{q}',R') &= \langle U(\sigma(\overrightarrow{p},\overrightarrow{q},R))\eta, U(\sigma(\overrightarrow{p}',\overrightarrow{q}',R'))\eta\rangle \\
&\quad\times |U(\sigma(\overrightarrow{p},\overrightarrow{q},R))\eta\rangle\langle U(\sigma(\overrightarrow{p}',\overrightarrow{q}',R'))\eta|,
\end{aligned} \tag{2.13}$$

and

$$\langle U(\sigma(\overrightarrow{p},\overrightarrow{q},R))\eta, U(\sigma(\overrightarrow{p}',\overrightarrow{q}',R'))\eta\rangle \neq 0 \tag{2.14}$$

for at least $(\overrightarrow{p},\overrightarrow{q},R)$ in a small neighborhood of $(\overrightarrow{p}',\overrightarrow{q}',R')$ by continuity of the representation. Alternatively, if $T^\eta(\overrightarrow{p},\overrightarrow{q},R)$ were projections that formed an orthogonal set, then we would have a nonseparable Hilbert space for $\mathcal{H}$. But $\mathcal{H}$ is separable.

Now let $\mathrm{d}\mu(\overrightarrow{p},\overrightarrow{q},R) = \mathrm{d}\lambda(\overrightarrow{q})\,\mathrm{d}\nu(\overrightarrow{p},R)$. Suppose f and h have marginal supports in space-like separated regions; i.e., the marginal support

$$\mathrm{supp}_q(f) \equiv \mathrm{supp}\left\{\iint_{(p,R)} f(\overrightarrow{p},\overrightarrow{q},R)\,\mathrm{d}\nu(\overrightarrow{p},R)\right\} \tag{2.15}$$

is spacelike separated from a similar expression in h. Then, analogously to property D above, one obtains

$$[A^\eta(f), A^\eta(h)] \neq 0 \tag{2.16}$$

for at least $\text{supp}_q(f)$ near $\text{supp}_q(h)$. In the following section, one finds that $[A^\eta(f), A^\eta(h)] \neq 0$ for all $f \neq ch$ (almost everywhere), c a constant. Thus one derives that these measurements 'in two space-like separated regions' never commute! Consequently, this theory of quantum measurements is non-local.

Finally, look at Bloch's paradox when one makes the assumption that by a measurement in a region Δ in one frame at time t, one means that one has a measurement operator $A^\eta(\chi_\Delta)$, and when measuring a normalized vector state $P_\psi = |\psi\rangle\langle\psi|$ in Δ, one will obtain the probability $\langle\psi|A^\eta(\chi_\Delta)|\psi\rangle$ at time t. Let us look at this when one measures $U(g)\psi$, $g \in \mathcal{P}$: one obtains the probability

$$\begin{aligned}\langle U(g)\psi|A^\eta(\chi_\Delta)|U(g)\psi\rangle &= \langle\psi|U(g)^\dagger A^\eta(\chi_\Delta)U(g)|\psi\rangle \\ &= \langle\psi|A^\eta(g^{-1}.\chi_\Delta)|\psi\rangle,\end{aligned} \tag{2.17}$$

the probability of measurement at the corresponding Lorentz transformed set! Similarly for measuring $A^\eta(f)$ on general state ρ.

There is no paradox of the character of Bloch's in this quantum mechanical measurement scheme because we are asking for the transition probabilities to be conserved between the wavefunction of the particle being observed and the particles $\{\eta_{p,q,R}\}$. Bloch's paradox arises if one forgets the fact that one must transform both the particle wavefunction and the wavefunction(s) with which the 'event' is taking place. Equivalently, Bloch's paradox may also be conceived as arising because you might have '$\psi(x) = \langle x, \psi\rangle$' with x being a point in configuration space and $\langle x|$ denoting, improperly, a delta function at x. Then under the Poincaré transformation g, $\psi \mapsto U(g)\psi$, and so $\psi(x)$ would become $[U(g)\psi](x)$. But being a square-integrable function in x, $\psi(x)$ has no meaning for any particular x! The view that $\psi(x)$ has meaning is a mixture of quantum mechanical theory (the ψ) and classical theory (the x); so, is a semi-classical view (which is wrong).

Moreover, it is commonly thought that one may choose to work in just the configuration space. Then one still would have properties A, B and D holding, with property C holding for just $U(\sigma(\vec{0}, \vec{q}, e))$. But then one would not have $U(g)$, for any $g \in \mathcal{P}$ appearing in (2.11) but rather only $U(\sigma(\vec{0}, \vec{q}, e))$, which do not generate the entire Poincaré group (and particularly the boosts) through $U(\sigma(\vec{0}, \vec{q}, e))U(h)$, $h \in H$. Hence one obtains the resulting loss of Poincaré covariance violating the usual set up of Bloch's paradox.

3. Informational completeness of the set of $A^\eta(f)$ s

One would have a poor measurement scheme if a 'complete' set of measurements of a state would not uniquely determine the state. In general recall

Definition 1 [8]. *Let ρ, ρ' be any density operators in Hilbert space $\mathcal{H}$. A set $\{A_x|x \in I\}$ of self-adjoint operators is informationally complete if $Tr(\rho A_x) = Tr(\rho' A_x)$ for all $x \in I$ implies that $\rho = \rho'$.*

One can show [3] that there is no 'complete set of commuting operators' in this sense. However, this assumption is hidden in Bloch's paper [2] (as it is in most papers that deal with such a set-up). Furthermore, any incomplete measurement would fail to satisfy condition (2) with which we first introduced Bloch's paradox.

In the present case, one wishes to investigate the informational completeness of a certain set associated with

$$\mathfrak{A}^+ \equiv \{A^\eta(f)|0 \leqslant f \leqslant 1, f\,\mu\text{-measurable}\}, \tag{3.1}$$

which one takes as the set of measurement operators for the simplest measurements. Using the linearity of the map $f \mapsto A^\eta(f)$, one takes

$$\mathfrak{A} = \mathfrak{A}^+ - \mathfrak{A}^+. \tag{3.2}$$

It has been shown [11] that, for any phase space representation, the (C*) algebra generated by $\mathfrak{A}$ by taking products (or rather the closure of this set) is informationally complete for η α-admissible and

$$\langle U(g)\eta, \eta\rangle \neq 0 \qquad \text{for almost every} \quad g \in \mathcal{P}. \tag{3.3}$$

Comparing with (2.14), one obtains complete nonlocality of the measurement operators $A^\eta(f)$; i.e., two nonzero measurement operators $A^\eta(f)$, $A^\eta(h)$ never commute unless $f = ch$ almost everywhere, c a constant, whether f and h have spacelike-separated marginal supports or not. One has a completely non-localized measurement theory. Furthermore [10], the set of measurement operators, $\mathfrak{A}^+$, does not contain any projection operators other than $\mathbf{1} = A^\eta(1)$ and $\mathbf{0} = A^\eta(0)$. Thus, one may not take (exactly) the projections onto the boxes X_i that appeared in the formulation of Bloch's paradox. Note that one may consider $A^\eta(\chi_\Delta)$ where $\Delta = X_i$ marginally, but that is not a projection operator. Having a normalized vector $\psi \in \mathcal{H}$ such that $\langle\psi, A^\eta(\chi_\Delta)\psi\rangle \simeq 1$ says just that the expected (average) value $(\overrightarrow{p}, \overrightarrow{q}, R)$ with respect to ψ is approximately in X_i. Similarly if χ_Δ is replaced by the fuzzy set function f and $|\psi\rangle\langle\psi|$ is replaced by a general state ρ.

In addition, there is no restriction on the size of the boxes employed, in spite of the popularity of making sure that the boxes are larger than the Compton wavelength, etc; these are boxes in the phase space of the *average values* $(\overrightarrow{p}, \overrightarrow{q}, R\overrightarrow{s_o})$, and if the volume of the boxes shrink below the quantum limit and go to zero, the transition probability goes smoothly to zero [9].

We remark that the set of conditions (2.2), (2.3) and (3.3) on the wavefunction η is always realizable. For example, consider the wavefunction for an electron in any of the basic states in an isolated hydrogen atom! We also note that everything we have said holds when the Poincaré group is replaced with the Heisenberg group, the Galilei group, the De Sitter groups or any other locally compact Lie group [9].

4. Summary

In the theory of quantum mechanics on phase space, quantum measurement is described by a covariant localization operator that is generated by a vector (in the Hilbert space of the particle) satisfying a certain type of admissibility condition. With an additional condition, the set of operators generated from the set of localization operators is informationally complete. Furthermore, this localization operator is optimal among all the potential modes of physical measurement. Using this form of localization operator and the unitarity of the Lorentz transformations, Bloch's paradox does not appear.

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